Math Meth Oper Res (2002) 56:269–286

Mathematical Methods
of Operations Research


# The Chi-compromise value for non-transferable utility games*

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Manuscript received: May 2001/Final version received: January 2002

**Abstract.** We introduce a compromise value for non-transferable utility games: the Chi-compromise value. It is closely related to the Compromise value introduced by Borm, Keiding, McLean, Oortwijn, and Tijs (1992), to the MC-value introduced by Otten, Borm, Peleg, and Tijs (1998), and to the $\Omega$-value introduced by Bergantiños, Casas-Méndez, and Vázquez-Brage (2000). The main difference being that the maximal aspiration a player may have in the game is his maximal (among all coalitions) marginal contribution. We show that it is well defined on the class of totally essential and non-level games. We propose an extensive-form game whose subgame perfect Nash equilibrium payoffs coincide with the Chi-compromise value.

**Key words:** NTU game, compromise value

## 1 Introduction

The purpose of this paper is to introduce a new compromise value for non-transferable utility games (NTU-games): the Chi-compromise value. As with all compromise values it chooses as the solution of the game the efficient vector lying in the segment between the vectors of maximal and minimal utilities that each player may expect to obtain; that is, it is a compromise between their

* We thank Carles Rafels, Howard Petith, and a referee for their helpful comments. Financial support from the Spanish Ministry of Education and Culture (through grants PB98-0870 and PB98-0613-C02-01), from the Xunta de Galicia (through grant PGIDT00PXI30001PN), and from the Departament d'Universitats, Recerca i Societat de la Informació de la Generalitat de Catalunya (through grant 2000SGR-0054) is gratefully acknowledged. The paper was partially written while Jordi Massó was visiting the Universidad Nacional de San Luis (Argentina). He acknowledges the hospitality of its Instituto de Matemática Aplicada and the financial support through a sabbatical fellowship from the Spanish Ministry of Education and Culture.

maximum and minimum aspirations. For pure bargaining problems (that is, situations where all agreements have to be unanimous) the Kalai-Smorodinsky solution (Kalai and Smorodinsky (1975)) is based on a compromise of this type. When partial agreements are possible and utility is transferable across players (that is, TU-games) we defined (Bergantiños and Massó (1996)) a compromise value called the Chi value. Our proposal here extends these two particular solutions to general problems where players may reach partial agreements and utility is not necessarily transferable (that is, NTU-games).

We propose as the maximum aspiration for a player in a game his maximal (among all coalitions) marginal contribution and as the minimum aspiration the maximum remainder he can obtain by going with a coalition of players and offering them their maximum aspirations. In non-level and totally essential NTU-games our proposed vectors of aspirations have the following three properties: (1) Giving players their maximum aspirations will always exhaust all possible gains from cooperation. (2) The vector of maximum aspirations is component-wise larger than the vector of minimal aspirations. (3) The minimum aspiration obtained in this rather indirect way coincides with the vector of individually rational payoffs. We find this last property interesting because it means that we have as a result that the minimum aspiration for each player in a game coincides with what he can obtain without any cooperation. It seems to us that this property may also be a good indication that the proposed maximum aspiration is meaningful.

The paper is organized as follows. Section 2 is a preliminary section which gives the main notation and concepts. Section 3 contains the definition of the Chi-compromise value; Propositions 1 and 2 and Corollary 1 which establish that properties (1), (2), and (3) above hold for non-level and totally essential NTU-games; the demonstration that the Chi-compromise value exists for all non-level and totally essential NTU-games; and finally, a number of examples which illustrate the new value. Section 4 provides two characterizations of the Chi-compromise value using the following axioms: Pareto Optimality, Covariance, Symmetry, and Restricted Monotonicity (or Strong Symmetry instead of Symmetry and Restricted Monotonicity). Section 5 proposes (as a generalization of Moulin (1984)'s implementation of the Kalai-Smorodinsky solution for pure bargaining problems) a non-cooperative extensive-form game whose subgame perfect equilibrium payoffs coincide with the Chi-compromise value. Section 6 proposes a different compromise value based on applying our Chi-value for TU-games to the characteristic function obtained by the classical $\lambda$-transfer approach. Section 7 concludes by comparing, briefly, our value with other well-known NTU-values.

## 2 Preliminaries

*Players* are the elements of a finite set $N = \{1, \dots, n\}$ where $n \geq 2$. A non-empty subset of players is called a *coalition*. We denote by $s$ the number of players of coalition $S$ and, abusing notation, by $i$ the singleton set $\{i\}$.

A (cooperative) game with *non-transferable utility* (NTU-game) is an ordered pair $(N, V)$ where $N = \{1, \dots, n\}$ is the set of *players* and $V$ is a mapping, called the *characteristic function*, which assigns to each non-empty coalition $S$ a non-empty subset of $\mathbb{R}^S$. By convenience, we set $V(\emptyset) = \emptyset$. The set $V(S)$ is interpreted as the collection of payoffs or utilities that members of $S$ can reach by cooperating among themselves. We will concentrate only on games with non-transferable utility having the standard properties that for each coalition $S$, the set $V(S)$ is closed, non-empty, and comprehensive (*i.e.*, $x \in V(S)$ and $y \leq x$ imply $y \in V(S)$). Given $x, y \in \mathbb{R}^K$, $y \leq x$ means $y_i \leq x_i$ for all $i \in K$ while $y < x$ means $y_i < x_i$ for all $i \in K$. Given $x \in \mathbb{R}^K$ and a coalition $S \subseteq K$, denote by $x_S$ the restriction of $x$ to the coordinates corresponding to the members of $S$; *i.e.*, $x_S = (x_i)_{i \in S}$. For each player $i \in N$ there exists a payoff $w_i \in \mathbb{R}$, called the *individually rational payoff*, such that $V(i) = \{x \in \mathbb{R} \mid x \leq w_i\}$. Also, for each coalition $S$, the set $V(S)_+ := \{x \in V(S) \mid x \geq w_S\}$ is bounded. We denote by $\mathbf{V}_N$ the class of games with non-transferable utility with set of players $N$.

We will often use the following properties of games with non-transferable utility.

**Definition 1.** *A game $(N, V)$ is* **non-level** *if for each coalition $S$ we have that for all $x, y \in V(S)_+$ such that $y \geq x \geq w_S$ and $x \neq y$ there exists $z \in V(S)$ with the property that $z > x$.*

**Definition 2.** *A game $(N, V)$ is* **totally essential** *if $w_S \in V(S)$ for all $S \subseteq N$.*

We denote by $\mathbf{C}_N$ the subclass of non-level and totally essential games with non-transferable utility.

A *solution* on a subclass of games $\mathbf{G}_N \subseteq \mathbf{V}_N$ is a function $\varphi : \mathbf{G}_N \to \mathbb{R}^N$ which assigns to each $(N, V) \in \mathbf{G}_N$ a vector $\varphi(N, V) \in V(N)$.

We will consider, and use as references, two special subclasses of games. A game $(N, V)$ has *transferable utility* if there is a real-valued function $v$ such that $V(S) = \{x \in \mathbb{R}^S \mid \sum_{i \in S} x_i \leq v(S)\}$; namely, each coalition $S$ can achieve a maximum level of utility $v(S)$ which can be distributed amongst its members in all possible ways. We denote by $\mathbf{v}_N$ the subclass of games with transferable utility with set of players $N$. A generic game with transferable utility will be denoted by $(N, v)$. A game $(N, V)$ is a *bargaining game* if it is totally essential and $V(S) = \{x \in \mathbb{R}^S \mid x \leq w_S\}$ for every coalition $S \neq N$; namely, there are gains from cooperation and they come only from unanimous agreements. We denote by $\mathbf{B}_N$ the subclass of bargaining games with set of players $N$. A generic bargaining game will be denoted by $(w, B)$, where $B$ stands for the set $V(N)$ and $w$ represents the disagreement point.

We are specially interested in extending two compromise solutions of these subclasses to games with non-transferable utility. The first one is the Kalai-Smorodinsky solution (Kalai and Smorodinsky (1975)) on bargaining games which represents an efficient compromise between the maximal aspiration of each player, compatible with individual rationality of the others, and the disagreement point. Formally, given $(w, B) \in \mathbf{B}_N$ define the *Kalai-Smorodinsky solution*, denoted by $KS(w, B)$, as follows: for all $i \in N$

$$KS_i(w, B) = \lambda M_i^{KS}(w, B) + (1 - \lambda) w_i,$$

where $M_i^{KS}(w, B) = \max\{x_i \in \mathbb{R} \mid (x_i, x_{N \setminus i}) \in B \text{ and } (x_i, x_{N \setminus i}) \geq w\}$ and $\lambda \in [0, 1]$ is such that $KS(w, B) \in P(B)$. $P(B)$ denotes the Pareto frontier of $B$. In general, given a set $A \subseteq \mathbb{R}^K$, the *Pareto frontier of $A$* is the set $P(A) = \{x \in A \mid \nexists y \in A \text{ satisfying } y \geq x, y \neq x\}$ and the *weak Pareto frontier of $A$* is the set $WP(A) = \{x \in A \mid \nexists y \in A \text{ satisfying } y > x\}$. By convenience, we set $P(\emptyset) = \emptyset$ and $WP(\emptyset) = \emptyset$. Given a set $A$ and a vector $y$ we say that $y$ is *undominated for* $A$ if $\nexists x \in A$ such that $x \geq y$ and $x \neq y$. Obviously, if $y \in V(S) \backslash P(V(S))$ then $y$ is dominated for $V(S)$.

The second one is the Chi value (Bergantiños and Massó (1996)) on the subclass of games with transferable utility. It is also based on selecting an efficient compromise between maximal and minimal aspirations of players. In this case, the maximal aspiration of a player is his largest marginal contribution while his minimal aspiration is the largest remainder he can obtain after conceding to the other players their maximal aspiration. Formally, let $(N, v)$ be a game with transferable utility. For each $i \in N$, define player $i$'s maximum aspiration in the game as

$$M_i^{\chi}(N, v) = \max_{S \subseteq N, i \in S} \{v(S) - v(S \backslash i)\}.$$

Given the vector $M^{\chi}(N, v)$ define player $i$'s minimum aspiration in the game as

$$m_i^{\chi}(N, v) = \max_{S \subseteq N, i \in S} \left\{ v(S) - \sum_{j \in S \backslash i} M_j^{\chi}(N, v) \right\}.$$

Define the *Chi value* on $\mathbf{v}_N$, denoted by $\chi(N, v)$, as the unique efficient vector in the lineal segment having as extreme points $m^{\chi}(N, v)$ and $M^{\chi}(N, v)$; that is,

$$\chi(N, v) = \gamma M^{\chi}(N, v) + (1 - \gamma) m^{\chi}(N, v),$$

where $\gamma \in [0, 1]$ is such that $\sum_{i \in N} \chi_i(N, v) = v(N)$. Bergantiños and Massó (1996) showed that the Chi value exists in the class of essential games with transferable utility (*i.e.*; $\sum_{i \in N} v(i) \leq v(N)$).

## 3 The Chi-compromise value

In this section we define and study a compromise value for NTU-games. Let $(N, V)$ be a game in $\mathbf{V}_N$. For each $i \in N$ define player $i$'s *maximum aspiration* in the game as

$$M_i^{\chi}(N, V) = \max_{S \subseteq N, i \in S} \{t \in \mathbb{R} \mid (t, x) \in V(S)_+, x \in P(V(S \backslash i))\}.$$

**Remark 1.** $M_i^{\chi}(N, V) \geq w_i$ (take $S = \{i\}$ and $t = w_i$).

We also have that $M_i^{\chi}(N, V) < +\infty$ because $V(S)_+$ is compact and $P(V(S \backslash i))$ is closed. Therefore, $M_i^{\chi}(N, V)$ is well defined for all $(N, V)$ in $\mathbf{V}_N$.

Given the vector $M^{\chi}(N, V)$ define player $i$'s *minimal aspiration* in the game as

$$m_i^{\chi}(N, V) = \max_{S \subseteq N, i \in S} \{t \in \mathbb{R} \mid (t, M_{S \backslash i}^{\chi}(N, V)) \in V(S)\}.$$

**Remark 2.** $m_i^{\chi}(N, V) \geq w_i$ (again, take $S = \{i\}$ and $t = w_i$).

Notice that for each $S$ containing $i$, the projection of $V(S)$ on $i$'s coordinate is closed and bounded above. Therefore the maximum defining $m_i^\chi(N, V)$ does exist for all $(N, V)$ in $\mathbf{V}_N$.

From now on, and when this does not lead to confusion, we will omit the reference to the game $(N, V)$ to denote the aspiration vectors $m^\chi$ and $M^\chi$.

Propositions 1 and 2 and Corollary 1 below state that the three important properties of the vectors of aspirations already explained in the Introduction hold for non-level and totally essential NTU-games. Proposition 1 says that, for every coalition $S$, the vector of maximum aspirations is undominated for $V(S)$.

**Proposition 1.** *For all* $(N, V) \in \mathbf{C}_N$ *and all* $S \subseteq N$

$$M_S^\chi \notin V(S) \backslash P(V(S)).$$

*Proof:* If $S$ has only one player the result holds. Suppose it is true when $S$ has at most $p-1$ players; we will show that the statement holds in the case of coalitions with $p$ players.

In order to get a contradiction assume that $S$ has $p$ players and $M_S^\chi \in V(S) \backslash P(V(S))$. Then, there exists $y_S \in V(S)$ such that $y_S \geq M_S^\chi$ and $i \in S$ with $y_i > M_i^\chi$. As $M_{S\backslash i}^\chi \notin V(S\backslash i) \backslash P(V(S\backslash i))$ (by the induction hypothesis) and $(N, V)$ is non-level we can find $x_{S\backslash i} \in P(V(S\backslash i))$ such that $x_{S\backslash i} \leq M_{S\backslash i}^\chi$. Then, by comprehensiveness, $(y_i, x_{S\backslash i}) \in V(S)$ and therefore $M_i^\chi \geq y_i > M_i^\chi$. ■

Proposition 2 below states that, for non-level and totally essential NTU-games, the vector of minimal aspirations coincides, as it should, with the vector of individually rational payoffs. But, again, notice that $m^\chi$ is obtained endogenously as the maximum reminder after giving to other players in the coalition their maximal aspirations. We interpret this property as an indication that our definition of maximal aspiration is sensible.

**Proposition 2.** *For all* $(N, V) \in \mathbf{C}_N$,

$$m^\chi = w.$$

*Proof:* From Remark 2 we already know that $m_i^\chi \geq w_i$. To see that $m_i^\chi \leq w_i$ it will be sufficient to show that $t \leq w_i$ for all $t \in \mathbb{R}$ and all $S \subseteq N$ such that $i \in S$ and $(t, M_{S\backslash i}^\chi) \in V(S)$. The proof is by induction on the number of players in the coalition $S$.

Assume that $S = \{i, j\}$. If $(t, M_j^\chi) \in V(\{i, j\})$ and $t > w_i$ then, by comprehensiveness of the game, $(x, M_j^\chi) \in V(\{i, j\})$ for all $x \leq t$, which is impossible by non-levelness of the game and the definition of $M_j^\chi$.

Assuming that the result is true if $S$ contains $p \geq 2$ players (the induction hypothesis), we will show that it is true for all coalitions with $p+1$ players. Let $S = \{i_1, \ldots, i_p, i\}$ be any set with $p+1$ players containing $i$ and assume that $(t, M_{S\backslash i}^\chi) \in V(S)$.

First we prove that if $t > w_i$ and $(t, M_{S\backslash i}^\chi) \in V(S)$ then $(t, M_{i_1}^\chi, \ldots, M_{i_{p-1}}^\chi) \in V(S\backslash i_p)$. Assume that $(t, M_{i_1}^\chi, \ldots, M_{i_{p-1}}^\chi) \notin V(S\backslash i_p)$. As $t > w_i$, $M_j^\chi \geq w_j$ for any $j \in N$, and $V$ is totally essential we can find $x \in P(V(S\backslash i_p))$ such that

$w_{S\setminus i_p} \leq x \leq (t, M^{\chi}_{S\setminus\{i,i_p\}})$. Therefore, $(x, M^{\chi}_{i_p}) \leq (t, M^{\chi}_{S\setminus i}) \in V(S)$ implying, by non-levelness of the game, that we can find a vector $y \in V(S)_+$ with the property that $y > (x, M^{\chi}_{i_p})$. Therefore, $y_{i_p} > M^{\chi}_{i_p}$ which contradicts the definition of $M^{\chi}_{i_p}$.

Now $t \leq w_i$ would follow by the induction hypothesis. ∎

Example 1 below shows that the conclusion of Proposition 2 does not hold for level NTU-games.

**Example 1.** Let $(N, V)$ be the NTU-game where $N = \{1, 2\}$, $w_1 = w_2 = 0$, and $V(N) = comp(conv(\{(1,1),(2,0)\}))$. In general, if $A \subseteq \mathbb{R}^K$, $comp(A)$ denotes the comprehensive hull of $A$ (*i.e.*, the smallest comprehensive set containing $A$) and $conv(A)$ the convex hull of $A$. The vector of maximum aspirations is $M^{\chi}(N, V) = (2, 1)$ and the vector of minimum aspirations is $m^{\chi}(N, V) = (1, 0)$ which for player 1 is strictly larger than $w_1 = 0$.

Corollary 1 explicitly states that for non-level and totally essential NTU-games the maximum aspiration is larger or equal to the minimum aspiration.

**Corollary 1.** *For all* $(N, V) \in \mathbf{C}_N$,

$$m^{\chi} \leq M^{\chi}.$$

*Proof:* It follows immediately from Proposition 2 and Remark 1. ∎

We can now define the Chi-compromise value as well as state the most important result of the paper which identifies a large class of games (non-level and totally essential) in which the Chi-compromise value does exist.

**Definition 3.** *The* **Chi-compromise value**, *denoted by* $\chi$, *is the unique efficient vector in the lineal segment having as extreme points* $m^{\chi}$ *and* $M^{\chi}$; *that is, for all* $(N, V) \in \mathbf{V}_N$,

$$\chi(N, V) = \gamma M^{\chi}(N, V) + (1 - \gamma) m^{\chi}(N, V),$$

*where* $\gamma$ *is the largest number in* $[0, 1]$ *satisfying* $\chi(N, V) \in P(V(N))$.

**Theorem 1.** *For all* $(N, V) \in \mathbf{C}_N$ *there exists* $\chi(N, V)$.

*Proof:* It follows by combining Propositions 1 and 2, and Corollary 1. ∎

**Remark 3.** It is straightforward to show that the Chi-compromise value coincides with the Kalai-Smorodinsky solution in bargaining problems and with the Chi value in TU-games.

We now compare more specifically our value with three compromise values in the literature: the Compromise value of Borm et al. (1992), the MC-value of Otten et al. (1998), and the $\Omega$-value of Bergantiños et al. (2000).

Given $(N, V) \in \mathbf{V}_N$, the Compromise value is defined as the unique vector on the lineal segment between $M^C(N, V)$ and $m^C(N, V)$ which lies in $V(N)$ and is closest to $M^C(N, V)$, where for any $i \in N$

$$M_i^C(N,V) = \sup\left\{ t \in \mathbb{R} \;\middle|\; \begin{array}{l} (t,x) \in V(N), x \notin V(N\backslash i)\backslash WP(V(N\backslash i)), \\ \text{and } x \geq w_{N\backslash i} \end{array} \right\}$$

and

$$m_i^C(N,V) = \max_{S\subseteq N, i\in S} \left\{ t \in \mathbb{R} \;\middle|\; \begin{array}{l} \exists x \in \mathbb{R}^{S\backslash i}, (t,x) \in V(S), \\ \text{and } x > M_{S\backslash i}^C(N,V) \end{array} \right\}.$$

The Compromise value exists for the class of compromise admissible NTU-games, defined as,

$$\mathbf{CA}_N = \left\{ (N,V) \in \mathbf{V}_N \;\middle|\; \begin{array}{l} m^C(N,V) \leq M^C(N,V), m^C(N,V) \in V(N), \\ \text{and } M^C(N,V) \notin V(N)\backslash WP(V(N)) \end{array} \right\}.$$

Borm et al. (1992) proved that for any $(N,V) \in \mathbf{V}_N$ and any $i \in N$, $m_i^C(N,V) \geq w_i$. Suppose that $(N,V)$ is non-level and hence $P(V(S)) = WP(V(S))$ for all $S \subseteq N$. Then, $m_i^C(N,V) \geq m_i^\chi(N,V)$. If $(t,x) \in V(N)$, $x \notin V(N\backslash i)\backslash WP(V(N\backslash i))$, and $x \geq w_{N\backslash i}$, by non-levelness, we can find $x' \in P(V(N\backslash i))$ such that $x' \leq x$ and hence $(t,x') \in V(N)_+$. Now, it is easy to conclude that $M_i^C(N,V) \leq M_i^\chi(N,V)$. Then, in the class of non-level NTU-games, $\mathbf{CA}_N \subset \mathbf{C}_N$; that is, if the Compromise value exists then the Chi-compromise value also exists.

Note that if in the definition of $M_i^\chi$ we change $x \in P(V(S\backslash i))$ to $x \in WP \cdot (V(S\backslash i))$ (denote this alternative maximum aspiration by $\overline{M}_i^\chi$) then it is straightforward to check that $\overline{M}_i^\chi(N,V) \geq M_i^C(N,V)$ for all NTU-games. Therefore, the corresponding Chi-compromise value using the $\overline{M}^\chi$ vector as maximum aspirations is defined whenever the Compromise value exists. However, it seems to us that it is more appropriate to obtain the maximum aspiration of a player $i$ in a coalition $S$ as the remainder assuming that the members of coalition $S\backslash i$ exhaust all their possible gains of cooperation by reaching Pareto (and not weakly Pareto) agreements.

The MC-value of Otten et al. (1998) is defined as the efficient outcome lying on the lineal segment between the vector of individually rational payoffs and a vector of maximum aspiration obtained by giving to each player the *sum* of *all* his marginal contributions in all possible orderings of the set of players. Since in many cases each component of this upper value vector may be unfeasible it seems difficult to justify it as a vector of maximal aspirations. Otten et al. (1998) showed that the MC-value is well defined in the class of monotonic, zero-normalized NTU-games, which is unrelated to the class of non-level and totally essential NTU-games.

The $\Omega$-value of Bergantiños et al. (2000) is defined as the efficient outcome lying on the lineal segment between the vector of individually rational payoffs and the vector of maximum aspirations $M^\Omega(N,V)$.

The vector of maximum aspirations and the $\Omega$-value are defined by using induction arguments. When $n = 2$ the vector of maximum aspirations $M^\Omega(N,V)$ is defined as in the Kalai-Smorodinsky bargaining solution ($M^{KS}$). Then, both solutions coincide when $n = 2$.

Suppose now that we have defined $M^\Omega$ and $\Omega$ when there are at most $n-1$ players.

Given $S \subset N$, $a_i(S) = \max\{t \in \mathbb{R} \mid (t, \Omega(S\backslash i, V|_{S\backslash i})) \in V(S)\}$ where $V|_{S\backslash i}$ denotes the restriction of $V$ to $S\backslash i$.

The maximum aspiration of player $i$ in the game $(N, V)$ is defined as $M_i^{\Omega}(N, V) = \max_{S \subset N, i \in S} a_i(S)$.

Bergantiños et al. (2000) prove that the $\Omega$-value exists for all non-level and superadditive NTU-games. $(N, V) \in \mathbf{V}_N$ is superadditive if for all $S, T \subset N$, $S \cap T = \emptyset$ then $V(S) \times V(T) \subset V(S \cup T)$. It is easy to check that the class of non-level and totally essential NTU-games, where the Chi-compromise value exists, is larger that the class of non-level and superadditive NTU-games, where the $\Omega$-value exists.

We end this section by calculating the Chi-compromise value in three well-known examples of NTU-games and comparing it with other proposed values.

**Example 2** (Roth (1980)). Let $(N, V)$ be an NTU-game such that $N = \{1, 2, 3\}$,

$$V(\{i\}) = \{x_i \in \mathbb{R}^{\{i\}} \mid x_i \leq 0\}, \quad \text{for } i \in N,$$

$$V(\{1,2\}) = \{(x_1, x_2) \in \mathbb{R}^{\{1,2\}} \mid (x_1, x_2) \leq (0.5, 0.5)\},$$

$$V(\{1,3\}) = \{(x_1, x_3) \in \mathbb{R}^{\{1,3\}} \mid (x_1, x_3) \leq (0.25, 0.75)\},$$

$$V(\{2,3\}) = \{(x_2, x_3) \in \mathbb{R}^{\{2,3\}} \mid (x_2, x_3) \leq (0.25, 0.75)\},$$

and

$$V(N) = \{x \in \mathbb{R}^N \mid \exists y \in conv\{(0.5, 0.5, 0), (0.25, 0, 0.75), (0, 0.25, 0.75)\}, \\ x \leq y\}.$$

For this example the Shapley-NTU value (Aumann (1985)) is (0.333, 0.333, 0.333), the Harsanyi-NTU value (Harsanyi (1963)) is (0.416, 0.416, 0.166), the Consistent value (Maschler and Owen (1989, 1992)) is (0.25, 0.25, 0.5), the MC-value coincides with the Shapley-NTU value, the Compromise value is (0.5, 0.5, 0), and the $\Omega$-value is (0.286, 0.286, 0.428).

Although the game does not satisfy non-levelness we can compute the Chi-compromise value, which is $(0.5, 0.5, 0)$, the unique Core outcome.

**Example 3** (Shafer (1980)). We present the modification of Shafer (1980)'s example as it was used in Hart and Kurz (1983). Consider the following exchange economy with three agents and two commodities. The initial commodity bundles of agents 1, 2, and 3 are

$$\omega^1 = (1 - \varepsilon, 0), \quad \omega^2 = (0, 1 - \varepsilon), \quad \text{and} \quad \omega^3 = (\varepsilon, \varepsilon),$$

where $0 \leq \varepsilon \leq \frac{1}{5}$, and their respective utility functions, $u_i$, are given by

$$u_1(y, z) = u_2(y, z) = \min\{y, z\}, \quad \text{and} \quad u_3(y, z) = \frac{y + z}{2}.$$

Following Shapley and Shubik (1969) the corresponding NTU-game $(N, V)$ is given by:

$$V(\{i\}) = \{x_i \in \mathbb{R}^{\{i\}} \mid x_i \leq 0\}, \quad \text{for } i = 1, 2,$$

$$V(\{3\}) = \{x_3 \in \mathbb{R}^{\{3\}} \mid x_3 \leq \varepsilon\},$$

$$V(\{1,2\}) = \{(x_1, x_2) \in \mathbb{R}^{\{1,2\}} \mid (x_1, x_2) \leq (1-\varepsilon, 1-\varepsilon), x_1 + x_2 \leq 1 - \varepsilon\},$$

$$V(\{1,3\}) = \left\{(x_1, x_3) \in \mathbb{R}^{\{1,3\}} \mid (x_1, x_3) \leq \left(\varepsilon, \frac{1+\varepsilon}{2}\right), x_1 + x_3 \leq \frac{1+\varepsilon}{2}\right\},$$

$$V(\{2,3\}) = \left\{(x_2, x_3) \in \mathbb{R}^{\{2,3\}} \mid (x_2, x_3) \leq \left(\varepsilon, \frac{1+\varepsilon}{2}\right), x_2 + x_3 \leq \frac{1+\varepsilon}{2}\right\},$$

and

$$V(N) = \{x \in \mathbb{R}^N \mid (x_1, x_2, x_3) \leq (1,1,1), x_1 + x_2 + x_3 \leq 1\}.$$

In this game the Shapley-NTU value is $\left(\frac{5-5\varepsilon}{12}, \frac{5-5\varepsilon}{12}, \frac{1+5\varepsilon}{6}\right)$, the Harsanyi-NTU value is $\left(\frac{3-5\varepsilon}{6}, \frac{3-5\varepsilon}{6}, \frac{5\varepsilon}{3}\right)$, the MC-value coincides with the Shapley-NTU value, the Compromise value is $\left(\frac{1-\varepsilon}{2}, \frac{1-\varepsilon}{2}, \varepsilon\right)$, and the $\Omega$-value is $\left(\frac{2-2\varepsilon}{5}, \frac{2-2\varepsilon}{5}, \frac{1+4\varepsilon}{5}\right)$.

The Chi-compromise value is $\left(\frac{2-2\varepsilon}{5-5\varepsilon}, \frac{2-2\varepsilon}{5-5\varepsilon}, \frac{1-\varepsilon}{5-5\varepsilon}\right)$.

**Example 4** (Owen (1972)). Let $(N, V)$ be an NTU-game such that $N = \{1, 2, 3\}$;

$$V(\{i\}) = \{x_i \in \mathbb{R}^{\{i\}} \mid x_i \leq 0\}, \quad \text{for } i \in N,$$

$$V(\{1,2\}) = \{(x_1, x_2) \in \mathbb{R}^{\{1,2\}} \mid x_1 + 4x_2 \leq 100, x_1 \leq 100, x_2 \leq 25\},$$

$$V(\{1,3\}) = \{(x_1, x_3) \in \mathbb{R}^{\{1,3\}} \mid x_1 \leq 0, x_3 \leq 0\},$$

$$V(\{2,3\}) = \{(x_2, x_3) \in \mathbb{R}^{\{2,3\}} \mid x_2 \leq 0, x_3 \leq 0\},$$

and

$$V(N) = \left\{x \in \mathbb{R}^N \mid \sum_{i \in N} x_i \leq 100;\ \forall i \in N, x_i \leq 100;\ \forall i, j \in N, x_i + x_j \leq 100\right\}.$$

In this example the Shapley-NTU value is $(50, 50, 0)$, the Harsanyi-NTU value is $(40, 40, 20)$, the Consistent value is $(50, 37.5, 12.5)$, the MC-value is $(50, 33.33, 16.67)$, the Compromise value is $(36.36, 36.36, 27.27)$, and the $\Omega$-value is $(42.1, 42.1, 15.7)$.

The Chi-compromise value is $(36.36, 36.36, 27.27)$.

## 4 Characterizations of the Chi-compromise value

In this section we study several properties of the Chi-compromise value. Moreover two characterizations of the Chi-compromise value are provided. To do that, let $\mathbf{G}_N \subseteq \mathbf{V}_N$ be an arbitrary subclass of NTU-games and let $\varphi$ be a solution on $\mathbf{G}_N$.

Pareto Optimality. *The solution $\varphi$ satisfies* **Pareto Optimality** *on $\mathbf{G}_N$ if $\varphi(N, V) \in P(V(N))$ for all $(N, V) \in \mathbf{G}_N$.*

Covariance. *The solution $\varphi$ satisfies* **Covariance** *on $\mathbf{G}_N$ if $\varphi(N, W) = \alpha * \varphi \cdot (N, V) + \beta$ whenever $(N, V), (N, W) \in \mathbf{G}_N$ are such that for all $S \subseteq N$, $W(S) = \alpha_S * V(S) + \beta_S$, where $\alpha_S * V(S) = \{(\alpha_i x_i)_{i \in S} \mid x_S \in V(S)\}$, $\alpha \in \mathbb{R}^N$, $\alpha > 0$ and $\beta \in \mathbb{R}^N$.*

Players $i$ and $j$ have a *symmetric position* in a game $(N, V)$ if (1) for $S \subseteq N\backslash\{i, j\}$, $x \in V(S \cup i)$ iff $y \in V(S \cup j)$ when $y_S = x_S$ and $y_j = x_i$ and (2) for $S \supseteq \{i, j\}$, $x \in S$ iff $y \in S$ when $y_{S\backslash\{i,j\}} = x_{S\backslash\{i,j\}}$, $y_i = x_j$, and $y_j = x_i$.

Symmetry. *The solution $\varphi$ satisfies* **Symmetry** *on $\mathbf{G}_N$ if $\varphi_i(N, V) = \varphi_j(N, V)$ whenever i and j have a symmetric position in the game $(N, V) \in \mathbf{G}_N$.*

Strong Symmetry. *The solution $\varphi$ satisfies* **Strong Symmetry** *on $\mathbf{G}_N$ if $\varphi_i(N, V) = \varphi_j(N, V)$ whenever $(N, V) \in \mathbf{G}_N$ is such that $w_i = w_j$ and $M_i^{\chi}(N, V) = M_j^{\chi}(N, V)$.*

Restricted Monotonicity. *The solution $\varphi$ satisfies* **Restricted Monotonicity** *on $\mathbf{G}_N$ if $\varphi(N, V) \leq \varphi(N, V')$ whenever $(N, V), (N, V') \in \mathbf{G}_N$ are such that $V(N) \subseteq V'(N)$, $w = w'$, and $M^{\chi}(N, V) = M^{\chi}(N, V')$.*

**Proposition 3.** *The Chi-compromise value satisfies Pareto Optimality, Covariance, Symmetry, Strong Symmetry, and Restricted Monotonicity on the class $\mathbf{C}_N$ of non-level and totally essential games with non-transferable utility.*

*Proof:* It is straightforward to check that the Chi-compromise value satisfies these five properties. ■

**Theorem 2.** *The Chi-compromise value is the unique solution on $\mathbf{C}_N$ satisfying Pareto Optimality, Covariance, Symmetry, and Restricted Monotonicity.*

*Proof:* We have just established in Proposition 3 that the Chi-compromise value satisfies the four properties.

We now prove uniqueness. Suppose $F$ is another solution satisfying the four properties. Assume that $w \notin P(V(N))$, otherwise the result is trivial.

First we prove that if $(N, V) \in \mathbf{C}_N$ and $w \notin P(V(N))$ then for all $i \in N$, $M_i^{\chi}(N, V) > w_i$. As $w \in V(N)\backslash P(V(N))$ and $(N, V)$ satisfies non-levelness there exists $x \in V(N)$, $x > w$. Given $i \in N$, as $V(\{i\}) = \{x \in \mathbb{R}^{\{i\}} \mid x \leq w_i\}$ we can find $S \subset N$ and $y \in V(S)_+$ such that $i \in S$, $V(S\backslash i)_+ = w_{S\backslash i}$, and $y > w_S$. This means that $M_i^{\chi}(N, V) > w_i$.

By Covariance it suffices to prove that $\chi(N, V) = F(N, V)$ when, for all $i \in N$, $w_i = 0$ and $M_i^{\chi}(N, V) = 1$.

Clearly, for all $i \in N$, the vector $c^i \in \mathbb{R}^N$ defined by $c^i_j = \chi_j(N, V) + \varepsilon$ if $i = j$ and $c^i_j = 0$ if $j \neq i$ belongs to $V(N)$ for $\varepsilon$ sufficiently small. The non-levelness ensures that $\varepsilon$ is strictly positive. Note that for all $i \in N$, $\chi_i(N, V) \leq 1$.

Let $(N, W)$ be such that for all $i \in N$

$$W(\{i\}) = \{x \in \mathbb{R}^{\{i\}} \mid x \leq 0\},$$

for all $S \subset N$ such that $2 \leq s \leq n-1$

$$W(S) = comp\left\{x_S \in \mathbb{R}^S \mid \forall i \in S, 0 \leq x_i \leq 1, \text{ and } \sum_{i \in S} x_i \leq 1\right\},$$

and

$$W(N) = comp(conv(\{c^i \in \mathbb{R}^N \mid i \in N\} \cup \chi(N, V))) \cap V(N).$$

Then $(N, W) \in \mathbf{C}_N$, $M^\chi_i(N, W) = 1$ for all $i \in N$, and $\chi(N, V) = \chi(N, W)$. By symmetry for all $i, j \in N$, $F_i(N, W) = F_j(N, W)$. Note that even though $W(N)$ is not necessarily a symmetric set, $(N, W)$ is a symmetric game. Therefore by Pareto Optimality, $F(N, W) = \chi(N, W)$. By Restricted Monotonicity $F(N, W) \leq F(N, V)$, which implies $\chi(N, V) \leq F(N, V)$. But since $\chi$ satisfies Pareto Optimality we can conclude that $\chi(N, V) = F(N, V)$. ■

**Theorem 3.** *The Chi-compromise value is the unique solution on* $\mathbf{C}_N$ *satisfying Pareto Optimality, Covariance, and Strong Symmetry.*

*Proof:* Proposition 3 establishes that the Chi-compromise value satisfies these properties.

We now prove uniqueness. Suppose $F$ is another solution satisfying these properties. Using similar arguments to those already used in the proof of Theorem 2 we can assume that for all $i \in N$, $M^\chi_i(N, V) > w_i$. By Covariance it suffices to prove that $\chi(N, V) = F(N, V)$ when, for all $i \in N$, $w_i = 0$ and $M^\chi_i(N, V) = 1$.

By Strong Symmetry, for all $i, j \in N$, $F_i(N, V) = F_j(N, V)$ and $\chi_i(N, V) = \chi_j(N, V)$. By Pareto Optimality, $F(N, V) = \chi(N, V)$. ■

Note that all axioms used in both characterizations are independent. The egalitarian solution defined by Kalai and Samet (1985) satisfies all five properties except Covariance. The solution $f^1$ defined as $f^1(N, V) = w$ for all $(N, V) \in \mathbf{C}_N$ satisfies all properties except Pareto Optimality. The solution $f^2$ defined as the Shapley value when $(N, V)$ is a totally essential TU-game and the Chi-compromise value in the rest of the class $\mathbf{C}_N$ satisfies all properties except Strong Symmetry and Restricted Monotonicity. The solution $f^3$ defined as $f^3_i(N, V) = w_i$ for $i \neq 1$ and $f^3_1(N, V) = \max\{t \in \mathbb{R} \mid (t, w_{N\setminus 1}) \in V(N)\}$, satisfies all properties except Symmetry.

These axiomatic characterizations can be extended in the following way. Theorem 2 is also true for the class of NTU-games for which the Chi-compromise value exists and the condition of non-levelness is satisfied only for the set $V(N)_+$. Theorem 3 is also true for the class of NTU-games where the Chi-compromise value exists.

Moreover, notice that in both characterizations the sets $V(S)$ need not be convex. While this is also possible in the characterization of the MC-value it is not the case in the characterization of the Compromise value where the set $V(N)$ has to be convex.

## 5 Implementation of the $\chi$-value

Following the Nash program, there is a long tradition of justifying axiomatic bargaining solutions by means of equilibria of a non-cooperative game associated to the original bargaining problem. Moulin (1984) exhibits an extensive-form game whose subgame perfect equilibria induce the Kalai-Smorodinsky solution. Here, and following the procedure used by Hart and Mas-Colell (1996) to obtain the Consistent value by extending the non-cooperative implementation of the Nash bargaining solution to NTU-games (which also coincides with the Shapley value for TU-games), we extend Moulin's implementation of the Kalai-Smorodinsky solution for bargaining problems to NTU-games (which also coincides with the Chi value for TU-games).

Given a NTU-game $(N, V)$, we define the non-cooperative $n$-person game $\Gamma(N, V)$ as follows:

- *Round* 0. Each player $i$ makes a bid $p_i$, where $0 < p_i \leq 1$, and they are renumbered in decreasing order of their bids, $p_1 \geq p_2 \geq \cdots \geq p_n$ (players with tied bids are ordered randomly among themselves).
- *Round* 1. Player 1 proposes a payoff vector $x = (x_1, \ldots, x_n) \in V(N)$ to the approval of player $n$, who can either accept or reject it. If player $n$ accepts $x$ the game proceeds to round 2.
  If player $n$ rejects $x$ then he must choose a pair $(S^n, x^n)$, where $S^n \subset N$, $n \in S^n$, and $x^n \in V(S^n)$. Player $n$ proposes to the other players of $S^n$ to cooperate with him to obtain a payoff of $x^n$. Players in $S^n$ may accept or reject $x^n$ but they are forced to accept it if there is no a payoff vector $y \in V(S^n \backslash n)$ such that $y > x^n_{S^n \backslash n}$. There are two cases to be considered:
  1. If all players in $S^n \backslash n$ accept $x^n$ then a lottery is held in which, with probability $p_1$ the agreement achieved by the players of $S^n$ is implemented (that is, every player $j \in S^n$ obtains $x^n_j$, except player 1, if $1 \in S^n$, who receives $w_1$). The exceptional treatment to player 1 (the proposer) is to dissuade him from putting forward unreasonable proposals that make unanimous agreement impossible. Players of $N \backslash S^n$ return to round 0 and continue to bargain among themselves. With probability $1 - p_1$ the bargaining procedure finishes and every player $i \in N$ obtains $w_i$.
  2. If any player of $S^n$ rejects $x^n$ then player $n$ is removed from the bargaining procedure; i.e., player $n$ obtains $w_n$. Let $R(n)$ be the set of players who rejected $(S^n, x^n)$ and $i_n$ the player of this set with the largest index. Player $i_n$ must propose a payoff vector $z \in V(S^n \backslash n)$ such that $z > x^n_{S^n \backslash n}$, which exists because $i_n$ rejected $x^n$. With probability $p_1$ the payoff vector $z$ is implemented (that is, every player $j \in S^n \backslash n$ obtains $z_j$) and the players of $N \backslash S^n$ return to round 0 and continue to bargain among themselves. With probability $1 - p_1$ the bargaining procedure finishes and every player $i \in N$ obtains $w_i$.

- *Round* 2. Player 1 proposes the payoff vector $x = (x_1, \ldots, x_n)$ to the approval of player $n-1$, who can either accept it or reject it. If he accepts it the game proceeds to round 3.
  If he rejects he must make a counteroffer $(S^{n-1}, x^{n-1})$, where $n-1 \in S^{n-1}$ and $x^{n-1} \in V(S^{n-1})$, and the game proceeds as in the previous round replacing the role of player $n$ by player $n-1$.
- *Rounds* $3, \ldots, n-1$ are similar to rounds 1 and 2 but now considering players $n-2, \ldots, 2$ instead of players $n$ and $n-1$.

**Remark 4.** Since the number of players is finite, the game $\Gamma(N, V)$ terminates in a finite number of steps.

**Remark 5.** Round 0 is the same than Round 0 in Moulin (1984). Rounds $1, 2, \ldots, n-1$ are similar to rounds $1, 2, \ldots, n-1$ of Moulin (1984). The difference is that in Moulin (1984), if some player rejects the initial offer he must make a counteroffer to the rest of the players, who can reject or accept it. If somebody rejects it the disagreement point is enforced. However, in our game the player who rejected the initial offer can make a proposal to some smaller coalition. This modification is necessary because in NTU-games partial agreements are also possible. Moreover, when we restrict our procedure to a non-cooperative game induced by a bargaining game it coincides, basically, with Moulin (1984). The only difference is that in Moulin (1984) when a player makes a counterproposal the rest of the players always can reject it (in such a case, all receive the disagreement point). However, in our game players can not reject an offer which gives them at least the disagreement point.

**Remark 6.** Bergantiños et al. (2000) also gives an implementation of the $\Omega$-value using a non-cooperative game which generalizes Moulin (1984). We now compare the non-cooperative game defined in this paper and the one described in Bergantiños et al. (2000). Round 0 is the same in both non-cooperative games. Round 1 is different in two aspects. First, in our case players in $S^n$ are forced to accept $x^n$ if there is no a payoff vector $y \in V(S^n \backslash n)$ such that $y > x^n_{S^n \backslash n}$; in Bergantiños et al. (2000) players in $S^n$ can reject any offer. Second, if $x^n$ is rejected, in our non-cooperative game player $i_n$ must propose a payoff vector $z \in V(S^n \backslash n)$ such that $z > x^n_{S^n \backslash n}$, with probability $p_1$ the payoff vector $z$ is implemented and with probability $1 - p_1$ every player $i \in S^n$ obtains $w_i$; in Bergantiños et al. (2000) if $x^n$ is rejected then with probability $p_1$ players in $S^n \backslash n$ return to Round 0 and continue to bargain among themselves and with probability $1 - p_1$ every player $i \in S^n$ obtains $w_i$. The same two differences (adjusted in the natural way) also apply to the remaining rounds $2, \ldots, n-1$.

We now present the main result of this section which says that the non-cooperative game described above implements in subgame perfect Nash equilibrium strategies the Chi-compromise value.

**Theorem 4.** *Let $(N, V)$ be a non-level and totally essential NTU-game. Then, the non-cooperative game $\Gamma(N, V)$ has subgame perfect Nash equilibria (SPNE). Moreover, the payoff received by the players in all of them is $\chi(N, V)$.*

*Proof:* Let $(N, V)$ be a non-level and totally essential NTU-game. The proof is by induction on the number of players.

*Case* $n = 2$*:* It is easy to check that the set of SPNE of $\Gamma(\{1,2\}, V)$ coincides with the set of SPNE of the game of auctioning fractions of dictatorship, Moulin (1984), applied to the bargaining problem $((w_1, w_2), V(\{1,2\}))$. Then, by Moulin (1984), this set is non-empty and the payoff received by the two players in all of these equilibria is the payoff vector $KS((w_1, w_2), V(\{1,2\}))$, which is equal to $\chi(\{1,2\}, V)$. Hence, the statement of Theorem 4 holds whenever $n = 2$.

Induction hypothesis: Assume that the statement of Theorem 4 holds when there are strictly less than $n$ players.

Now, the proof that the statement of Theorem 4 is also true when there are $n$ players is based on Lemmas 1 and 2 below.

**Lemma 1.** *The set of SPNE of $\Gamma(N, V)$ is non-empty.*

*Proof of Lemma 1:* Let $p \in (0,1]$ be such that $\chi(N, V) = pM^{\chi}(N, V) + (1-p) \cdot m^{\chi}(N, V)$. The proof will consist of exhibiting a SPNE strategy profile $\sigma$.

*Definition of* $\sigma$: In round 0 each player $i$ submits a bid $p_i$ equal to $p$. The proposer, now player 1, proposes the vector $x = p_1 M^{\chi}(N, V) + (1-p_1)m^{\chi} \cdot (N, V)$. Every player $i \neq 1$ accepts the proposal $x$ of player 1 if and only if $x_j \geq p_1 M_j^{\chi}(N, V) + (1-p_1)m_j^{\chi}(N, V)$ for all $j \neq 1$. After rejecting $x$, player $i$ would propose $(S^i, x^i)$, where $S^i$ is the coalition that maximizes the reminder in the definition of $M_i^{\chi}(N, V)$; i.e., $x^i_{S^i \backslash i} \in P(V(S^i \backslash i))$ and $x^i_i = M_i^{\chi}(N, V)$. Players in $S^i \backslash i$ will accept any offer $y \in V(S^i)$ if and only if $y_{S^i \backslash i} \in P(V(S^i \backslash i))$. If the procedure goes back to Round 0 then, there is, at most, $n-1$ players. Hence, define $\sigma$ in these subgames as the behavior prescribed by an arbitrary SPNE strategy of the game with at most $n-1$ players, whose existence is guaranteed by the induction hypothesis.

Notice that the play prescribed by $\sigma$ is that the selected player (all of them with equal probability) proposes $\chi(N, V)$ and the rest accept it. Hence, the expected payoff induced by $\sigma$ in $\Gamma(N, V)$ is the Chi-compromise value of $(N, V)$.

To prove that $\sigma$ is an SPNE we have to show that no player, in any of its information sets, has incentives to deviate from $\sigma$.

First, if the game goes back to Round 0, by the definition of $\sigma$, no player has a profitable deviation.

Second, assume player $i$ rejected the initial offer of player 1 and proposed, according with $\sigma$, $(S^i, x^i)$. Then, all players in $S^i \backslash i$ are forced to accept it since there is no $z \in V(S^i \backslash i)$ such that $z_j > x^i_j$ for all $j \in S^i \backslash i$ because $x^i_{S^i \backslash i} \in P(V(S^i \backslash i))$.

Third, player $i$ has no profitable deviation from proposing $(S^i, x^i)$, which is what specifies $\sigma$ after he rejects an initial offer. To see it, suppose that player $i$ proposes any $(\hat{S}^i, \hat{x}^i)$ with the property that $\hat{x}^i_{\hat{S}^i \backslash i} \in V(\hat{S}^i \backslash i) \backslash P(V(\hat{S}^i \backslash i))$. Then, at least one player in $\hat{S}^i \backslash i$ will reject it and player $i$ will get $w_i$. Therefore, if player $i$ wants to obtain more than $w_i$ he must offer an acceptable proposal; that is, a pair $(\bar{S}^i, \bar{x}^i)$ with the property that $\bar{x}^i_{\bar{S}^i \backslash i} \in P(V(\bar{S}^i \backslash i))$, otherwise, at least one player will reject it. Among all of these pairs, to offer the pair $(S^i, x^i)$ specified by $\sigma$ is the best, which gives to player $i$ the payoff of $M_i^\chi(N, V)$ with probability $p$ and $w_i$ with probability $1 - p$; *i.e.*, $\chi_i(N, V)$.

As a consequence of the last two arguments (second and third) we can conclude that no player has incentives to reject the initial proposal $\chi(N, V)$.

Fourth, we show that player 1 does not get a strictly higher payoff by proposing $x \neq \chi(N, V)$. Suppose that $x_{N\backslash 1} \geq \chi_{N\backslash 1}(N, V)$, then the rest of the players will accept $x$; hence, player 1 gets $x_1 \leq \chi_1(N, V)$ because $x \in V(N)$ and $\chi(N, V) \in P(V(N))$. Suppose now that there exists $i \neq 1$ with $x_i < \chi_i \cdot (N, V)$. Then, $x$ will be rejected by player $n$, who will propose, according to $\sigma$, the pair $(S^n, x^n)$, which will be accepted by the members of $S^n$ since there is no $y \in V(S^n\backslash n)$ such that $y > x^n_{S^n\backslash n}$. Now we distinguish two cases:

- If $1 \in S^n$ player 1 gets $w_1$, which is not larger than $\chi_1(N, V)$.
- If $1 \in N\backslash S^n$ player 1 gets, by the induction hypothesis, $\chi_1(N\backslash S^n, V|_{N\backslash S^n})$ with probability $p$ and $w_1$ with probability $1 - p$. Taking into account that $\chi_1(N\backslash S^n, V|_{N\backslash S^n}) \leq M_1^\chi(N\backslash S^n, V|_{N\backslash S^n}) \leq M_1^\chi(N, V)$ we conclude that (also in this case) player 1 cannot get a strictly larger payoff than $\chi_1(N, V)$.

Finally, we show that, at stage 0, to make a bid different from $p$ is not a profitable deviation. Suppose that player $i$ bids $p_i < p$, which implies that $i$ is not the initial proposer. If he rejects $\chi(N, V)$ then, as we saw before, he obtains at most $\chi_i(N, V)$. Suppose now that player $i$ bids $p_i > p$. Then, he becomes player 1 and must make an offer $x \in V(N)$. If there exists a player $j \neq 1$ such that $x_j < p_1 M_j^\chi(N, V) + (1 - p_1)w_j$, $x$ will be rejected and using similar arguments to those used before we can conclude that player 1 gets at most $\chi_1(N, V)$. If $x_{N\backslash 1} \geq \chi_{N\backslash 1}(N, V)$, $x$ will be accepted but since $x \in V(N)$ and $\chi(N, V) \in P(V(N))$ we conclude that $x_1 < \chi_1(N, V)$. Therefore, $\sigma$ is an SPNE of $\Gamma(N, V)$. □

**Lemma 2.** *In any SPNE of $\Gamma(N, V)$ any player $i$ has an expected payoff of at least $\chi_i(N, V)$.*

*Proof of Lemma 2:* First we prove that if player $i \neq 1$ rejects the offer of player 1 and the players of $N\backslash i$ are playing according to an SPNE then player $i$ gets $p_1 M_i^\chi(N, V) + (1 - p_1)w_i$. Suppose that player $i$ proposes $(S^i, x^i)$. Using similar arguments to those already used in the proof of Lemma 1 we can conclude that player $i$ has to propose $(S^i, x^i)$ as in $\sigma$ and players in $S^i$ will accept it, which means that player $i$ gets $M_i^\chi(N, V)$ with probability $p_1$ and $w_i$ with probability $(1 - p_1)$. We can also show that player $i$ can not obtain strictly more.

We now prove that in any SPNE any player $i \neq 1$ receives at least $p_1 M_i^\chi \cdot (N, V) + (1 - p_1)w_i$. We prove it by finding a deviation of player $i$ which gives him $p_1 M_i^\chi(N, V) + (1 - p_1)w_i$. Assume that player $i \neq 1$ makes a bid $p_i' < p_n$ instead of $p_i$. Then player $i$ (although he becomes player $n$ we will still refer to him as player $i$) is the first who answers the offer of player 1. If player $i$ rejects it we proved before that he will receive $p_1 M_i^\chi(N, V) + (1 - p_1)w_i$.

Now, to get a contradiction, suppose that there exists an SPNE where a player, $i$, receives a payoff $y_i < \chi_i(N, V)$. We study several cases:

1. $p_1 > p$. Then, player $i$ cannot be a responder; otherwise, we have just proved that his payoff would be at least $p_1 M_i^\chi + (1 - p_1)w_i$, which is strictly

larger than $pM_i^{\chi} + (1-p)w_i = \chi_i(N, V)$. Therefore, $i$ is the proposer, and hence, $i = 1$. Now we distinguish two cases:

- $p_2 \geq p$. Suppose that player 1 makes a bid $p_1' < p_n$. Then, if he rejects the offer of player 2 he obtains $p_2 M_1^{\chi}(N, V) + (1-p_2)w_1 \geq \chi_1(N, V)$, as we saw at the beginning of the proof. But this is a contradiction because we found a deviation of player 1 $(p_1' < p_1)$ which strictly improves his payoff.
- $p_2 < p$. Suppose player 1 makes a bid $p$ and offers $x \in P(V(N))$ such that $x_1 > y_1$ and for all $i \neq 1$, $x_i = \chi_i(N, V) + \varepsilon$ where $\varepsilon > 0$ is chosen in an appropriate and obvious way. The players of $N \backslash 1$ will accept $x$ (if player $i$ rejects $x$ we already proved that he would obtain $pM_i^{\chi}(N, V) + (1-p)w_i = \chi_i(N, V)$). Then, player 1 can strictly improve his payoff by bidding $p$ instead of $p_1$, which is a contradiction.

2. $p_1 < p$. Suppose player $i$ makes a bid $p$. Then, he becomes the winner of the auction because $p_1$ was the largest bid (again, we will still refer to him as player $i$). Moreover, assume that $i$ offers any $x \in P(V(N))$ with the property that $x_i > y_i$ and for all $j \neq i$, $x_j = \chi_j(N, V) + \varepsilon$ where $\varepsilon > 0$ is chosen in an appropriate way. Players in $N \backslash i$ will accept $x$ (we have already proved that if player $j$ rejects $x$ he would obtain $pM_j^{\chi}(N, V) + (1-p)w_j = \chi_j(N, V)$). Then, player $i$ can strictly improve his payoff by bidding $p$ instead of $p_i$, which is a contradiction.
3. $p_1 = p$. We study several cases:
   - $i \neq 1$. If player $i$ makes a bid $p_i' < p_n$ with similar arguments to the case $p_1 > p$ and $p_2 \geq p$ we obtain that player $i$ can strictly improve his payoff.
   - $i = 1$ and $p_2 = p$. If player 1 makes a bid $p_1' < p_n$ with similar arguments to the case $p_1 > p$ and $p_2 \geq p$ we obtain that player 1 can strictly improve his payoff.
   - $i = 1$ and $p_2 < p$. Again, with similar arguments to those used in the case $p_1 > p$ and $p_2 < p$ we can conclude that player 1 can strictly improve his payoff. ■

The proof of Theorem 4 finishes by noticing that by the definition of $\Gamma(N, V)$ and Lemma 2 in any SPNE each player $i$ has an expected payoff of $\chi_i(N, V)$ because $\chi(N, V) \in P(V(N))$. ■

## 6 The Lambda-transfer Chi-value

Shapley (1969) defined the family of $\lambda$-transfer TU-games corresponding to an NTU-game. Using this family of games, and their corresponding Shapley values, he defined the NTU-Shapley value. We proceed in the same way using our Chi value for TU-games instead of the Shapley value.

Define $\Delta^N = \{\lambda \in \mathbb{R}^N \mid \sum_{i \in N} \lambda_i = 1 \text{ and } \lambda_i \geq 0 \text{ for all } i\}$ as the $n$-dimensional unit simplex. Given an NTU-game $(N, V)$ we say that the vector $\lambda \in \Delta^N$ is *feasible* if $\sup\left\{\sum_{i \in S} \lambda_i x_i \mid x \in V(S)\right\} < \infty$ for all $S \subseteq N$. For each feasible vector $\lambda \in \Delta^N$ we define the TU-game $(N, v^{\lambda})$ by associating with each coalition $S \subseteq N$ the number $v^{\lambda}(S) = \sup\left\{\sum_{i \in S} \lambda_i x_i \mid x \in V(S)\right\}$.

**Definition 4.** *The* **Lambda-transfer Chi-value** *on* $\mathbf{V}_N$*, denoted by* $\chi^\Lambda$*, associates to each* $(N, V) \in \mathbf{V}_N$ *the set*

$$\chi^\Lambda(N, V) = \{x \in V(N) \mid \lambda * x \geq \chi(N, v^\lambda) \text{ for some } \lambda \in \Delta^N \text{ feasible}\}.$$

Before stating a result establishing sufficient conditions under which the Lambda-transfer Chi-value set is non-empty we need to define two standard properties of NTU-games.

**Definition 5.** *An NTU-game* $(N, V)$ *is* **compactly generated** *if for all* $S \subseteq N$ *there exists a compact set* $K_S \subset \mathbb{R}^S$ *with the property that* $V(S) = \{x \in \mathbb{R}^S \mid x \leq y$ *for some* $y \in K_S\}$*. An NTU-game* $(N, V)$ *is* **convex** *if for all* $S \subseteq N$ *the set* $V(S)$ *is convex.*

**Theorem 5.** *Let* $(N, V)$ *be a totally essential, compactly generated, and convex NTU-game. Then,* $\chi^\Lambda(N, V) \neq \emptyset$*.*

*Proof:* First, we will show that if the NTU-game $(N, V)$ is totally essential then for any feasible $\lambda \in \Delta^N$ the TU-game $(N, v^\lambda)$ is essential. Consider any $i \in N$. By definition $v^\lambda(i) = \lambda_i w_i$. Moreover, as $(N, V)$ is totally essential,

$$\begin{aligned} v^\lambda(N) &= \sup\left\{\sum_{i \in N} \lambda_i x_i \mid x \in V(N)\right\} \\ &\geq \sum_{i \in N} \lambda_i w_i \\ &= \sum_{i \in N} v^\lambda(\{i\}), \end{aligned}$$

which means that the TU-game $(N, v^\lambda)$ is essential.

The non-emptiness of the set $\chi^\Lambda(N, V)$ follows using a fixed-point argument similar to that of Shapley (1969). ■

The game of Example 2 illustrates the fact that, in general, the Chi-compromise value and the Lambda-transfer Chi-value may be different. After a simple, but very tedious computation, it is possible to see that $\chi^\Lambda(N, V) = (0.33, 0.33, 0.33)$ while $\chi(N, V) = (0.5, 0.5, 0)$.

## 7 Concluding remarks

Before finishing this paper we would like to briefly compare our proposal with other NTU-values. As with all compromise values it is easier to compute than the Shapley, Harsanyi, and the Consistent values. However, the Shapley and Harsanyi values have nice characterizations, while those of *all* compromise values including ours are *ad hoc* (in the sense that the vectors of maximum and minimum aspirations are used in the definitions of some of the key axioms); on the contrast, to our knowledge the Consistent value has yet to be fully characterized (Maschler and Owen (1989) characterize it for the class of hyperplane games). Except for the Compromise value and the $\Omega$-value, whose existence is guaranteed only for games with non-empty cores (a proper subclass of compromise admissible NTU-games) and superadditive games respectively, the existence of all other NTU-values is guaranteed for classes of games which are relatively larger than these and unrelated to each other. Finally, to our knowledge, only the Consistent value (Hart and Mas-Colell (1996)), the $\Omega$-value, and our Chi-compromise value have been shown to be implementable by extensive-form games.

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